DIFFERENTIALLY PRIVATE QUERY RELEASE THROUGH ADAPTIVE PROJECTION

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ABSTRACT

We propose, implement, and evaluate a new algorithm for releasing answers to very large numbers of statistical queries like k-way marginals, subject to differential privacy. Our algorithm makes adaptive use of a continuous relaxation of the *Projection Mechanism*, which answers queries on the private dataset using simple perturbation, and then attempts to find the synthetic dataset that most closely matches the noisy answers. We use a continuous relaxation of the synthetic dataset domain which makes the projection loss differentiable, and allows us to use efficient ML optimization techniques and tooling. Rather than answering all queries up front, we make judicious use of our privacy budget by iteratively and adaptively finding queries for which our (relaxed) synthetic data has high error, and then repeating the projection. We perform extensive experimental evaluations across a range of parameters and datasets, and find that our method outperforms existing algorithms in many cases, especially when the privacy budget is small or the query class is large.

1 INTRODUCTION

A basic problem in differential privacy is to accurately answer a large number m of statistical queries (also known as *linear* and *counting* queries), which have the form, “how many people in private dataset D have property P?” Marginal queries (also known as *conjunctions*) are one of the most useful and most studied special cases. The simplest technique for answering such queries is to compute each answer on the private dataset, and then perturb them with independent Gaussian noise. This simple technique is useful for answering small numbers of queries. But it has been known since Blum et al. (2008) that *in principle*, it is possible to privately and accurately answer very large classes of queries (of size exponential in n), and that an attractive way of doing so is to encode the answers in a *synthetic dataset*. Synthetic datasets have several advantages: most basically, they are a concise way of representing the answers to large numbers of queries. But they also permit one to evaluate queries *other* than those that have been explicitly answered by the mechanism, and to take advantage of *generalization*. Unfortunately, it is also known that improving on the error of the simple Gaussian perturbation technique is computationally hard in the worst case (Ullman, 2016). Moreover, constructing synthetic datasets is hard even when it would be possible to provide accurate answers with simple perturbation (Ullman & Vadhan, 2011) for simple classes of queries such as the set of all $\binom{d}{2}$ marginal queries restricted to 2 out of d binary features (so-called 2-way marginals).

Nevertheless, recent years have seen a resurgence of interest in private synthetic data generation and large-scale private queries due to the importance of the problem. These approaches offer provable privacy guarantees, but have run-time and accuracy properties that must be evaluated empirically. This is the literature to which our work contributes.

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1.1 Our Contributions

Our starting point is the (computationally inefficient) *projection mechanism* of Nikolov et al. (2013), which is informally described as follows. We begin with a dataset $D \in \mathcal{X}^n$. First, the values of each of the m queries of interest q_i are computed on the private dataset: $a = q(D) \in [0,1]^m$. Next, a privacy preserving vector of noisy answers $\hat{a} \in \mathbb{R}^m$ is computed using simple Gaussian perturbation. Finally, the vector of noisy answers $\hat{a}$ is *projected* into the set of answer vectors that are consistent with some dataset to obtain a final vector of answers a' — i.e., the projection guarantees that $a' = q(D')$ for *some* $D' \in \mathcal{X}^n$. This corresponds to solving the optimization problem of finding the synthetic dataset $D' \in \mathcal{X}^n$ that minimizes error $||q(D') - \hat{a}||_2$. This is known to be a near optimal mechanism for answering statistical queries (Nikolov et al., 2013) but for most data and query classes, the projection step corresponds to a difficult discrete optimization problem. We remark that the main purpose of the projection is not (only) to construct a synthetic dataset, but to improve accuracy.

Our core algorithm is based on a continuous relaxation of this projection step. This allows us to deploy first-order optimization methods, which empirically work very well despite the non-convexity of the problem. A further feature of this approach is that we can take advantage of sophisticated existing tooling for continuous optimization — including autodifferentiation (to allow us to easily handle many different query classes) and GPU acceleration, which has been advanced by a decade of research in deep learning. This is in contrast to related approaches like Gaboardi et al. (2014); Vietri et al. (2020) which use integer program solvers and often require designing custom integer programs for optimizing over each new class of queries. We then extend our core algorithm by giving an adaptive variant that is able to make better use of its privacy budget, by taking advantage of generalization properties. Rather than answering *all* of the queries up front, we start by answering a small number of queries, and then project them onto a vector of answers consistent with a relaxed synthetic dataset — i.e., a dataset in a larger domain than the original data — but one that still allows us to evaluate queries. At the next round, we use a private selection mechanism to find a small number of additional queries on which our current synthetic dataset performs poorly; we answer those queries, find a new synthetic dataset via our continuous projection, and then repeat. If the queries we have answered are highly accurate, then we are often able to find synthetic data representing the original data well after only having explicitly answered a very small number of them (i.e., we *generalize* well to new queries). This forms a virtuous cycle, because if we only need to explicitly answer a very small number of queries, we can answer them highly accurately with our privacy budget.

We evaluate our algorithm on several datasets, comparing it to two state-of-the-art algorithms from the literature: the high dimensional matrix mechanism from McKenna et al. (2018), and the FEM ("Follow-the-Perturbed-Leader with Exponential Mechanism") mechanism from Vietri et al. (2020). We find that our algorithm generally outperforms both of these approaches across a range of parameters — especially in the important and challenging high privacy and large query workload regimes.

2 The Relaxed Adaptive Projection (RAP) Mechanism

We here introduce the "Relaxed Adaptive Projection" (RAP) mechanism (Algorithm 2), which has three hyper-parameters: the *number of adaptive rounds* T, the *number of queries per round* K, and the *size of the (relaxed) synthetic dataset* n'. In the simplest case, when $T = 1$ and $K = m$, we recover the natural relaxation of the projection mechanism: (1) We evaluate each query $q_i \in Q$ on D using the Gaussian mechanism to obtain a noisy answer $\hat{a}_i$, and (2) Find a *relaxed* synthetic dataset $D' \in X^r$ whose equivalent extended differentiable query values are closest to $\hat{a}$ in ℓ_2 norm: $D' = \arg\min_{D' \in (\mathcal{X}^r)^{n'}} ||\hat{q}(D') - \hat{a}||_2$. Because step 2 is now optimizing a continuous, differentiable function over a continuous space (of dimension $d' \cdot n'$, we can use existing tool kits for performing the optimization – for example, we can use auto-differentiation tools, and optimizers like Adam Kingma & Ba (2015). Here n' is a hyperparameter that we can choose to trade off the expressivity of the synthetic data with the running-time of the optimization: If we choose $n' = n$, then we are assured that it is possible to express D exactly in our relaxed domain: as we choose smaller values of n', we introduce a source of representation error, but decrease the dimensionality of the optimization problem in our projection step, and hence improve the run-time of the algorithm. In this simple case, we can recover an accuracy theorem by leveraging the results of Nikolov et al. (2013):

In the general case, our algorithm runs in T rounds: After each round t, we have answered some *subset* of the queries $Q_S \subseteq Q$, and perform a projection only with respect to the queries in Q_S for which we have estimates, obtaining an intermediate relaxed synthetic dataset D'_t. At the next round, we augment Q_S with K additional queries q_i from $Q \setminus Q_S$ chosen (using report noisy max) to maximize the disparity $|q_i(D'_t) - q_i(D)|$. We then repeat the projection. In total, this algorithm only explicitly answers $T \cdot K$ queries, which might be $\ll m$. But by selectively answering queries for which the consistency constraints imposed by the projection with respect to previous queries have not correctly fixed, we aim to expend our privacy budget more wisely. Adaptively answering a small number of "hard" queries has its roots in a long theoretical line of work (Roth & Roughgarden, 2010; Hardt & Rothblum, 2010; Gupta et al., 2012).

Algorithm 1 Relaxed Projection (RP)

```
**Input:** A vector of differentiable queries $q : \mathcal{X}^r \to \mathbb{R}^{m'}$, a vector of target answers $\hat{a} \in \mathbb{R}^{m'}$, and an initial dataset $D' \in (\mathcal{X}^r)^{n'}$.
Use any differentiable optimization technique (Stochastic Gradient Descent, Adam, etc.) to attempt to find:
    $D_S = \arg \min_{D' \in (\mathcal{X}^r)^{n'}} ||q(D') - \hat{a}||_2^2$
Output $D_S$.
```

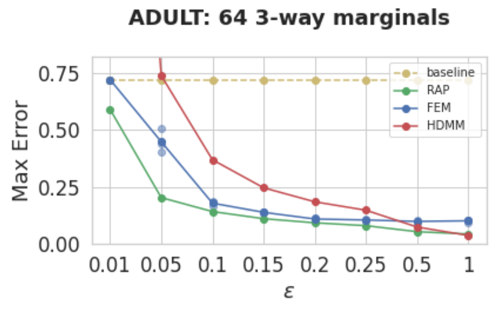
Algorithm 2 Relaxed Adaptive Projection (RAP)

```
**Input:** A dataset $D$, a collection of $m$ statistical queries $Q$, a "queries per round" parameter $K \leq m$, a "number of iterations" parameter $T \leq m/K$, a synthetic dataset size $n'$, and differential privacy parameters $\epsilon, \delta$.
Let $\rho$ be such that:
    $\epsilon = \rho + 2\sqrt{\rho \log(1/\delta)}$
**if** $T = 1$ **then**
  **for** $i = 1$ to $m$ **do**
    Let $\hat{a}_i = G(D, q_i, \rho/m)$.
  **end for**
  Randomly initialize $D' \in (\mathcal{X}^r)^{n'}$.
  Output $D' = RP(q, \hat{a}, D')$.
**else**
  Let $Q_S = \emptyset$ and $D'_0 \in (\mathcal{X}^r)^{n'}$ be an arbitrary initialization.
  **for** $t = 1$ to $T$ **do**
    **for** $k = 1$ to $K$ **do**
      Define $\hat{q}^{Q \setminus Q_S}(x) = (\hat{q}_i(x) : q_i \in Q \setminus Q_S)$ where $\hat{q}_i$ is an equivalent extended differentiable query for $q_i$.
      Let $q_i = RNM(D, q^{Q \setminus Q_S}, q^{Q \setminus Q_S}(D'_{t-1}), \frac{\rho}{2T \cdot K})$.
      Let $Q_S = Q_S \cup \{q_i\}$.
      Let $\hat{a}_i = G(D, q_i, \frac{\rho}{2T \cdot K})$.
    **end for**
    Define $q^{Q_S}(x) = (q_i(x) : q_i \in Q_S)$ and $\hat{a} = \{\hat{a}_i : q_i \in Q_S\}$ where $\hat{q}_i$ is an equivalent extended differentiable query for $q_i$. Let $D'_t = RP(q^{Q_S}, \hat{a}, D'_{t-1})$.
  **end for**
  Output $D'_T$.
**end if**
```

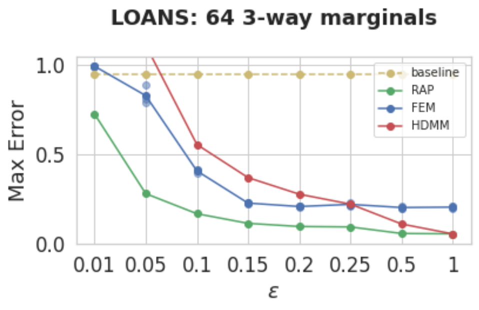
3 Experimental Results

We evaluate the algorithm on the two datasets used by Vietri et al. (2020) in their evaluation: ADULT and LOANS (Dua & Graff, 2017). Just as in Vietri et al. (2020), both datasets are transformed so that all features are categorical — real valued features are first bucketed into a finite number of categories. The algorithms are then run on a one-hot encoding of the discrete features, as we described in Section A.2. To ensure consistency, we use the pre-processed data exactly as it appears in their repository for Vietri et al. (2020). See Table A1 for a summary of the datasets.

To compare algorithms, we mirror the evaluation in Vietri et al. (2020) and focus our experiments on answering 3-way and 5-way marginals. We compare to the FEM algorithm from Vietri et al. (2020) and the High Dimensional Matrix Mechanism (HDMM) from McKenna et al. (2018). We use the maximum error between answers to queries on the synthetic data and the correct answers on the real data across queries ($\max_i |q_i(D') - q_i(D)|$) as a performance measure. For calibration, we also report a naive baseline corresponding to the error obtained by answering every query with "0". Error above this naive baseline is uninteresting. For all experiments, we fix the privacy parameter δ to $\frac{1}{n^2}$, where n is the number of records in the dataset, and vary ϵ as reported.



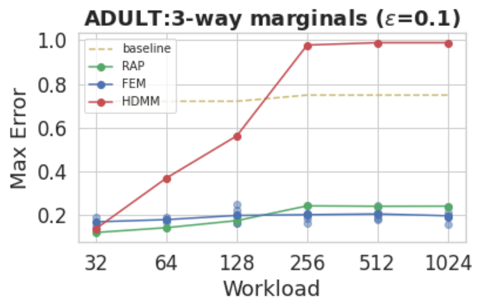
(a) ADULT dataset on 3-way marginals



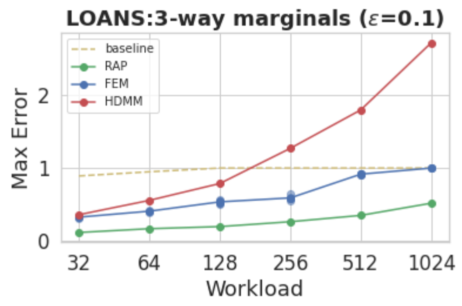
(b) LOANS dataset on 3-way marginals

Figure 1: Max-error for 3-way marginal queries on different privacy levels. The number of marginals is fixed at 64.

In Fig. 1 (see Fig. A2 for 5-way marginals) we show how our performance scales with the privacy budget ϵ for a fixed number of marginals. Fig. 2 (see Fig. A3 for 5-way queries) shows our performance for a fixed privacy budget as we increase the number of marginals being preserved. We significantly outperform both FEM and HDMM in a large majority of comparisons considered, and performance is particularly strong in the important high-privacy and high workload regimes (i.e., when ϵ is small and m is large).



(a) ADULT dataset on 3-way marginals



(b) LOANS dataset on 3-way marginals

Figure 2: Max error for increasing number of 3-way marginal queries with $\epsilon = 0.1$

4 Conclusion

We have presented a new, extensible method for privately answering large numbers of statistical queries, and producing a form of synthetic data consistent with those queries. Our method relies on a continuous, differentiable relaxation of the projection mechanism, which allows us to use existing powerful tooling developed for deep learning. We demonstrate on a series of experiments that our method generally out-performs existing techniques across a wide range of parameters, especially in the "high privacy" (i.e., small ϵ) and large workload regimes.

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A Appendix

A.1 Preliminaries

A.1.1 Statistical Queries and Synthetic Data

Let $\mathcal{X}$ be a data domain. In this paper, we will focus on data points containing d categorical features: i.e. $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_d$, where each $\mathcal{X}_i$ is a set of t_i categories. A *dataset* (which we will denote by D) consists of a multiset of n points from $\mathcal{X}$: $D \in \mathcal{X}^n$.

Definition A.1 (Statistical Query Kearns (1998))**.** A *statistical query* (also known as a *linear query* or *counting query*) is defined by a function $q_i : \mathcal{X} \to [0, 1]$. Given a dataset D, we will abuse notation and write $q_i(D)$ to denote the average value of the function q_i on D:

$$q_i(D) = \frac{1}{n} \sum_{x \in D} q_i(x)$$

Given a collection of m statistical queries $\{q_i\}_{i=1}^m$, we write $q(D) \in [0, 1]^m$ to denote the vector of values $q(D) = (q_1(D), \ldots, q_m(D))$.

An important type of statistical query is a k-way marginal, which counts the number of data points $x \in D$ that have a particular realization of feature values for some subset of k features.[1]

Definition A.2. A k-way marginal query is defined by a subset $S \subseteq [d]$ of $|S| = k$ features, together with a particular value for each of the features $y \in \prod_{i \in S} \mathcal{X}_i$. Given such a pair (S, y), let $\mathcal{X}(S, y) = \{x \in \mathcal{X} : x_i = y_i \ \ \forall i \in S\}$ denote the set of points that match the feature value y_i for each of the k features in S. The corresponding statistical query $q_{S,y}$ is defined as:

$$q_{S,y}(x) = \mathbb{1}(x \in \mathcal{X}(S, y))$$

Observe that for each collection of features (*marginal*) S, there are $\prod_{i \in S} |\mathcal{X}_i|$ many queries.

Given a set of m statistical queries q, we will be interested in vectors of answers $a' \in [0, 1]^m$ that represent their answers on D *accurately*:

Definition A.3. Given a dataset D, a collection of m statistical queries represented as $q : \mathcal{X}^n \to [0, 1]^m$, and a vector of estimated answers $a' \in [0, 1]^m$, we say that a' has ℓ_∞ or *max* error α if $\max_{i \in [m]} |q_i(D) - a'_i| \leq \alpha$.

In this paper we will represent vectors of estimated answers a' *implicitly* using some data structure D' on which we can evaluate queries, and will write $q(D')$ for a'. If $D' \in \mathcal{X}^*$, then we refer to D' as a *synthetic dataset* — but we will also make use of D' lying in continuous relaxations of $\mathcal{X}^n$ (and will define how query evaluation applies to such "relaxed datasets").

A.1.2 Differential Privacy

Two datasets $D, D' \in \mathcal{X}^n$ are said to be *neighboring* if they differ in at most one data point. We will be interested in *randomized algorithms* $\mathcal{A} : \mathcal{X}^n \to R$ (where R can be an arbitrary range).

Definition A.4 (Differential Privacy Dwork et al. (2006a;b))**.** A randomized algorithm $\mathcal{A} : \mathcal{X}^n \to R$ is (ϵ, δ) differentially private if for all pairs of neighboring datasets $D, D' \in \mathcal{X}^n$ and for all measurable $S \subseteq R$:

$$\Pr[\mathcal{A}(D) \in S] \leq \exp(\epsilon) \Pr[\mathcal{A}(D') \in S] + \delta.$$

If $\delta = 0$ we say that $\mathcal{A}$ is ϵ-differentially private.

Differential privacy is not convenient for tightly handling the degradation of parameters under composition, and so as a tool for our analysis, we use the related notion of (zero) Concentrated Differential Privacy:

[1]We define marginals for datasets with discrete features. In our experimental results we encode continuous features as discrete by standard binning techniques.

Definition A.5 (Zero Concentrated Differential Privacy Bun & Steinke (2016))**.** An algorithm $\mathcal{A} : \mathcal{X}^n \rightarrow R$ satisfies ρ-zero Concentrated Differential Privacy (zCDP) if for all pairs of neighboring datasets $D, D' \in \mathcal{X}^n$, and for all $\alpha \in (0, \infty)$:

$$\mathbb{D}_\alpha(\mathcal{A}(D), \mathcal{A}(D')) \leq \rho\alpha$$

where $\mathbb{D}_\alpha(\mathcal{A}(D), \mathcal{A}(D'))$ denotes the α-Renyi divergence between the distributions $\mathcal{A}(D)$ and $\mathcal{A}(D')$.

zCDP enjoys clean composition and postprocessing properties:

Lemma A.6 (Composition Bun & Steinke (2016))**.** Let $\mathcal{A}_1 : \mathcal{X}^n \rightarrow R_1$ be ρ_1-zCDP. Let $\mathcal{A}_2 : \mathcal{X}^n \times R_1 \rightarrow R_2$ be such that $\mathcal{A}_2(\cdot, r)$ is ρ_2-zCDP for every $r \in R_1$. Then the algorithm $\mathcal{A}(D)$ that computes $r_1 = \mathcal{A}_1(D)$, $r_2 = \mathcal{A}_2(D, r_1)$ and outputs (r_1, r_2) satisfies $(\rho_1 + \rho_2)$-zCDP.

Lemma A.7 (Post Processing Bun & Steinke (2016))**.** Let $\mathcal{A} : \mathcal{X}^n \rightarrow R_1$ be ρ-zCDP, and let $f : R_1 \rightarrow R_2$ be an arbitrary randomized mapping. Then $f \circ \mathcal{A}$ is also ρ-zCDP.

Together, these lemmas mean that we can construct zCDP mechanisms by modularly combining zCDP sub-routines. Finally, we can relate differential privacy with zCDP:

Lemma A.8 (Conversions Bun & Steinke (2016))**.**

1. If $\mathcal{A}$ is ϵ-differentially private, it satisfies $(\frac{1}{2}\epsilon^2)$-zCDP.
2. If $\mathcal{A}$ is ρ-zCDP, then for any $\delta > 0$, it satisfies $(\rho + 2\sqrt{\rho \log(1/\delta)}, \delta)$-differential privacy.

We will make use of two basic primitives from differential privacy, which we introduce here in the context of statistical queries. The first is the Gaussian mechanism.

Definition A.9 (Gaussian Mechanism)**.** The Gaussian mechanism $G(D, q_i, \rho)$ takes as input a dataset $D \in \mathcal{X}^n$, a statistical query q_i, and a zCDP parameter ρ. It outputs $a_i = q_i(D) + Z$, where $Z \sim N(0, \sigma^2)$, where $N(0, \sigma^2)$ is the Gaussian distribution with mean 0 and variance $\sigma^2 = \frac{1}{2n^2\rho}$.

Lemma A.10 (Bun & Steinke (2016))**.** For any statistical query q_i and parameter $\rho > 0$, the Gaussian mechanism $G(\cdot, q_i, \rho)$ satisfies ρ-zCDP.

The second is a simple private "selection" mechanism called report noisy max — we define a special case here, tailored to our use of it.

Definition A.11 (Report Noisy Max)**.** The "Report Noisy Max" mechanism $RNM(D, q, a, \rho)$ takes as input a dataset $D \in \mathcal{X}^n$, a vector of m statistical queries q, a vector of m conjectured query answers a, and a zCDP parameter ρ. It outputs the index of the query with highest noisy error estimate. Specifically, it outputs $i^* = \arg\max_{i \in [m]}(|q_i(D) - a_i| + Z_i)$ where each $Z_i \sim \text{Lap}\left(\frac{2}{n\sqrt{2\rho}}\right)$.

Lemma A.12. For any vector of statistical queries q, vector of conjectured answers a, and zCDP parameter ρ, the Report Noisy Max mechanism $RNM(\cdot, q, a, \rho)$ satisfies ρ-zCDP.

Proof. Definition A.11 defines a special case of the "report-noisy-max" mechanism which is proven to be ϵ-differentially private for $\epsilon = \sqrt{2\rho}$ in Dwork & Roth (2014). The ρ-zCDP bound then follows from Lemma A.8. $\square$

A.2 Relaxing the Projection Mechanism

The projection mechanism of Nikolov et al. (2013) can be described simply in our language. Given a collection of m statistical queries q and zCDP parameter ρ, it consists of two steps:

1. For each i, evaluate q_i on D using the Gaussian mechanism: $\hat{a}_i = G(D, q_i, \rho/m)$.
2. Find the synthetic dataset[2] D' whose query values are closest to $\hat{a}$ in ℓ_2 norm — i.e., let $D' = \arg\min_{D' \in \mathcal{X}^*} ||q(D') - \hat{a}||_2$.

[2]In fact, in Nikolov et al. (2013), the projection is onto a set of datasets that allows datapoints to have positive or negative weights — but their analysis also applies to projections onto the set of synthetic datasets in our sense. A statement of this can be found as Lemma 5.3 in Błasiok et al. (2019).

The output of the mechanism is the synthetic dataset D', which implicitly encodes the answer vector $a' = q(D')$. Because the perturbation in Step 1 is Gaussian, and the projection is with respect to the ℓ_2 norm, D' is the maximum likelihood estimator for the dataset D given the noisy statistics $\hat{a}$. The projection also serves to enforce consistency constraints across all query answers, which perhaps counter-intuitively, is accuracy-improving. For intuition, the reader can consider the case in which all queries q_i are identical: in this case, the scale of the initial Gaussian noise is $\Omega(\sqrt{m}/n)$, which is sub-optimal, because the single query of interest could have been privately answered with noise scaling only as $O(1/n)$. But the effect of the projection will be similar to *averaging* all of the perturbed answers $\hat{a}_i$, because $q_i(D')$ will be constrained to take a fixed value across all i (since the queries are identical), and the mean of a vector of noisy estimates minimizes the Euclidean distance to those estimates. This has the effect of averaging out much of the noise, recovering error $O(1/n)$. The projection mechanism is easily seen to be ρ-zCDP — the m applications of (ρ/m)-zCDP instantiations of the Gaussian mechanism in Step 1 compose to satisfy ρ-zCDP by the composition guarantee of zCDP (Lemma A.6), and Step 2 is a post-processing operation, and so by Lemma A.7 does not increase the privacy cost. This mechanism is nearly optimal amongst the class of all differentially private mechanisms, as measured by ℓ_2 error, in the worst case over the choice of statistical queries Nikolov et al. (2013). Unfortunately, Step 2 is in general an intractable computation, since it is a minimization of a non-convex and non-differentiable objective over an exponentially large discrete space. The first idea that goes into our algorithm (Algorithm 1) is to relax the space of datasets $\mathcal{X}^n$ to be a continuous space, and to generalize the statistical queries q_i to be differentiable over this space. Doing so allows us to apply powerful GPU-accelerated tools for differentiable optimization to the projection step 2.

From Categorical to Real Valued Features Our first step is to embed categorical features into *binary* features using a one-hot encoding. This corresponds to replacing each categorical feature $\mathcal{X}_i$ with t_i binary features $\mathcal{X}_i^1 \times \ldots \times \mathcal{X}_i^{t_i} = \{0,1\}^{t_i}$, for each $x \in \mathcal{X}$. Exactly one of these new t_i binary features corresponding to categorical feature i is set to 1 for any particular data point $x \in \mathcal{X}$: If $x_i = v_j$ for some $v_j \in \mathcal{X}_i$, then we set $\mathcal{X}_i^j = 1$ and $\mathcal{X}_i^{j'} = 0$ for all $j' \neq j$. Let $d' = \sum_{i=1}^{d} t_i$ be the dimension of a feature vector that has been encoded using this one-hot encoding. Under this encoding, the datapoints x are embedded in the binary feature space $\{0,1\}^{d'}$. We will aim to construct synthetic data that lies in a continuous relaxation of this binary feature space. For example, choosing $\mathcal{X}^r = [0,1]^{d'}$ is natural. In our experiments, we choose $\mathcal{X}^r = [-1,1]^{d'}$, which empirically leads to an easier optimization problem.

Let $h : \mathcal{X} \to \{0,1\}^{d'}$ represent the function that maps a $x \in \mathcal{X}$ to its one-hot encoding. We abuse notation and for a dataset $D \in \mathcal{X}^n$, write $h(D)$ to denote the one-hot encoding of every $x \in D$.

From Discrete to Differentiable Queries Consider a marginal query $q_{S,y} : \mathcal{X} \to \{0,1\}$ defined by some $S \subseteq [d]$ and $y \in \prod_{i \in S} \mathcal{X}_i$. Such a query can be evaluated on a vector of categorical features $x \in \mathcal{X}$ in our original domain. Our goal is to construct an *equivalent extended differentiable query* $\hat{q}_{S,y} : \mathcal{X}^r \to \mathbb{R}$ that has two properties:

Definition A.13 (Equivalent Extended Differentiable Query)**.** Given a statistical query $q_i : \mathcal{X} \to [0,1]$, we say that $\hat{q}_i : \mathcal{X}^r \to \mathbb{R}$ is an extended differentiable query that is equivalent to q_i if it satisfies the following two properties:

1. $\hat{q}_i$ is differentiable over $\mathcal{X}^r$ — i.e. for every $x \in \mathcal{X}^r$, $\nabla q_i(x)$ is defined, and
2. $\hat{q}_i$ agrees with q_i on every feature vector that results from a one-hot encoding. In other words, for every $x \in \mathcal{X}$: $q_i(x) = \hat{q}_i(h(x))$.

We will want to give equivalent extended differentiable queries for the class of k-way marginal queries. Towards this end, we define a product query:

Definition A.14. Given a subset of features $T \subseteq [d']$, the product query $q_T : \mathcal{X}^r \to \mathbb{R}$ is defined as: $q_T(x) = \prod_{i \in T} x_i$.

By construction, product queries satisfy the first requirement for being extended differentiable queries: they are defined over the entire relaxed feature space $\mathcal{X}^r$, and are differentiable (since they are monomials over a real valued vector space). It remains to observe that for every marginal query $q_{S,y}$, there is an equivalent product query $\hat{q}_{S,y}$ that takes value $q_{S,y}(x)$ on the one-hot encoding $h(x)$ of x for every x.

Lemma A.15. Every k-way marginal query has an equivalent extended differentiable query in the class of product queries. In other words, for every k-way marginal query $q_{S,y} : \mathcal{X}^n \to \{0,1\}$, there is a corresponding product query $\hat{q}_{S,y} = q_T(y) : \mathcal{X}^r \to \mathbb{R}$ with $|T| = k$ such that for every $x \in \mathcal{X}$: $q_{S,y}(x) = q_T(h(x))$.

Proof. We construct T in the straightforward way: for every $i \in S$, we include in T the coordinate corresponding to $y_i \in \mathcal{X}_i$. Now consider any x such that $q_{S,y}(x) = 1$. It must be that for every $i \in S$, $x_i = y_i$. By construction, the product $q_T(h(x)) = \prod_{j \in T} h(x)_j = 1$ because all terms in the product evaluate to 1. Similarly, if $q_{S,y}(x) = 0$, then it must be that for at least one coordinate $j \in T$, $h(x)_j = 0$, and so $q_T(h(x)) = \prod_{j \in T} h(x)_j = 0$. □

A.3 Theorems

A.3.1 Accuracy Theorem

Theorem A.16. *Fix privacy parameters $\epsilon, \delta > 0$, a synthetic dataset size n', and any set of m k-way product queries q. If the minimization in the projection step is solved exactly, then the average error for the RAP mechanism when $T = 1$ and $K = m$ can be bounded as:*

$$\sqrt{\frac{1}{m}||q(D) - q(D')||_2^2} \leq$$
$$O\left(\frac{(d'(\log k + \log n') + \log(1/\beta)\ln(1/\delta))^{1/4}}{\sqrt{\epsilon n}} + \frac{\sqrt{\log k}}{\sqrt{n'}}\right)$$

with probability $1 - \beta$ over the realization of the Gaussian noise.

Proof. We reduce to the (unrelaxed) projection mechanism, which has the following guarantee proven by Nikolov et al. (2013): for any dataset D consisting of n elements from a *finite* data universe $\mathcal{X}$, and for any set of m statistical queries q, the projection mechanism results in a dataset D' such that: $\sqrt{\frac{1}{m}||q(D') - q(D)||_2^2} \leq \alpha$ for

$$\alpha = O\left(\frac{(\ln(|\mathcal{X}|/\beta)\ln(1/\delta))^{1/4}}{\sqrt{\epsilon n}}\right).$$

Consider a finite data universe $\mathcal{X}^\eta = \{0, \eta, 2\eta, \ldots, 1\}^{d'}$ for some discretization parameter $0 < \eta < 1/k$. Given a dataset $D' \in \mathcal{X}^r$, let $D'_\eta \in \mathcal{X}^\eta$ be the dataset that results from "snapping" each real-valued $x \in D$ to its closest discrete valued point $x_\eta \in \mathcal{X}^r$. Observe that by construction, $||x - x(\eta)||_\infty \leq \eta$, and as a result, for k-way product query q_i, we have $|q_i(D') - q_i(D'_\eta)| \leq O(\eta k)$. Now let $\hat{D}' = \arg\min_{\hat{D}' \in (\mathcal{X}^r)^*} ||a - q(\hat{D}')||$ and $D'' = \arg\min_{D'' \in (\mathcal{X}^\eta)^*} ||a - q(D'')||$. From above, we know that $\sqrt{\frac{1}{m}||q(D'') - q(\hat{D}')||} \leq O(\eta k)$, and hence from an application of the triangle inequality, we have that $\sqrt{\frac{1}{m}||q(D) - q(\hat{D}')||_2^2} \leq O\left(\frac{(\ln(|\mathcal{X}^\eta|/\beta)\ln(1/\delta))^{1/4}}{\sqrt{\epsilon n}} + \eta k\right)$. Finally, for any dataset $\hat{D}' \in (\mathcal{X}^r)^*$, there exists a dataset $D' \in (\mathcal{X}^r)^{n'}$ such that $\sqrt{\frac{1}{m}||q(D') - q(\hat{D}')||_2^2} \leq O(\frac{\sqrt{\log k}}{\sqrt{n'}})$ (This follows from a sampling argument, and is proven formally in Blum et al. (2008).) Hence, a final application of the triangle inequality yields:

$$\sqrt{\frac{1}{m}||q(D) - q(D')||_2^2} \leq$$
$$O\left(\frac{(\ln(|\mathcal{X}^\eta|/\beta)\ln(1/\delta))^{1/4}}{\sqrt{\epsilon n}} + \eta k + \frac{\sqrt{\log k}}{\sqrt{n'}}\right)$$

Choosing $\eta = \frac{\sqrt{\log k}}{k\sqrt{n'}}$ and noting that $|\mathcal{X}^\eta| = (\frac{1}{\eta})^{d'}$ yields the bound in our theorem.

□

This is an "oracle efficient" accuracy theorem in the style of Gaboardi et al. (2014); Vietri et al. (2020); Neel et al. (2020) in the sense that it assumes that our heuristic optimization succeeds (note that this assumption is not needed for the privacy of our algorithm, which we establish in Theorem A.17). Compared to the accuracy theorem for the FEM algorithm proven in Vietri et al. (2020), our theorem improves by a factor of $\sqrt{d'}$.

A.3.2 Privacy Theorem

Theorem A.17. *For any query class Q, any set of parameters K, T, n', and any privacy parameters $\epsilon, \delta > 0$, the RAP mechanism $RAP(\cdot, Q, K, T, n', \epsilon, \delta)$ (Algorithm 2) is (ϵ, δ)-differentially private.*

Proof. The privacy of Algorithm 2 follows straightforwardly from the tools we introduced in Section A.1. First consider the case of $T = 1$. The algorithm makes m calls to the Gaussian mechanism, each of each satisfies ρ/m-zCDP by construction and Lemma A.10. In combination, this satisfies ρ-zCDP by the composition Lemma (Lemma A.6). It then makes a call to the relaxed projection algorithm RP, which is a postprocessing of the Gaussian mechanism, and hence does not increase the zCDP parameter, by Lemma A.7. Hence the algorithm is ρ-zCDP, and by our choice of ρ and Lemma A.8, satisfies (ϵ, δ) differential privacy.

Now consider the case of $T > 1$. Each iteration of the inner loop makes one call to report noisy max, and one call to the Gaussian mechanism. By construction and by Lemmas A.10 and A.12, each of these calls satisfies $\frac{\rho}{2TK}$-zCDP, and together by the composition Lemma A.6, satisfy $\frac{\rho}{TK}$-zCDP. The algorithm then makes a call to the relaxed projection algorithm RP, which is a post-processing of the composition of the Gaussian mechanism with report noisy max, and so does not increase the zCDP parameter by Lemma A.7. The inner loop runs $T \cdot K$ times, and so the entire algorithm satisfies ρ-zCDP by the composition Lemma A.6. By our choice of ρ and Lemma A.8, our algorithm satisfies (ϵ, δ) differential privacy as desired. $\square$

A.4 Implementation and Hyperparameters

We implement Algorithm 2 in Python Van Rossum & Drake (2009), using the JAX library Bradbury et al. (2018) for auto-differentiation of queries and the Adam optimizer Kingma & Ba (2015) for the call to RP (Algorithm 1). Fig. A1 contains a Jax code snippet, which computes 3-way product queries on a dataset D. A benefit of using JAX (or other packages with autodifferentiation capabilities) is that to instantiate the algorithm for a new query class, all that is required is to write a new python function which computes queries in the class — we do not need to perform any other reasoning about the class. In contrast, approaches like Gaboardi et al. (2014); Vietri et al. (2020) require deriving an integer program to *optimize* over each new class of interest. This makes our method more easily extensible.

JAX also has the advantages of being open source and able to take advantage of GPU acceleration. We run our experiments for Algorithm 2 on an EC2 p2.xlarge instance (1 GPU, 4 CPUs, 61 GB RAM). For FEM we use the code from the authors of Vietri et al. (2020) available at `https://github.com/giusevtr/fem`, using the hyperparameters given in their tables 2 and 3 for the experimental results we report in Figures A2 and A3, respectively. Their code requires the Gurobi integer program solver; we were able to obtain a license to Gurobi for a personal computer, but not for EC2 instances, and so we run FEM on a 2019 16" MacBook Pro (6 CPUs, 16GB RAM) (Gurobi does not support GPU acceleration) — as a result we do not report timing comparisons. We remark that an advantage of our approach is that it can leverage the robust open-source tooling (like JAX and Adam) that has been developed for deep learning, to allow us to easily take advantage of large-scale distributed GPU accelerated computation. For HDMM's implementation we use the code available at `https://github.com/ryan112358/private-pgm/blob/master/examples/hdmm.py`, which we also run on a MacBook Pro.

For most experiments, we set the size of the synthetic data $n' = 1000$ — significantly smaller than n for both of our datasets (see Table A1).

A.5 Selecting Marginals

As in Vietri et al. (2020), given k, we select a number of marginals S (i.e., subsets of categorical features), referred to as the *workload*, at random, and then enumerate all queries consistent with the selected marginals (i.e., we enumerate all $y \in \prod_{i \in S} \mathcal{X}_i$). For each experiment, we fix the query selection process and random seed so that all algorithms in our comparisons are evaluated on exactly the same set of queries.

A.6 Additional Tables for Experiments

Dataset	Records	Features	Transformed Binary Features
ADULT	48842	15	588
LOANS	42535	48	4427

Table A1: Datasets. Each dataset starts with the given number of original (categorical and real valued) features. After our transformation, it is encoded as a dataset with a larger number of binary features.

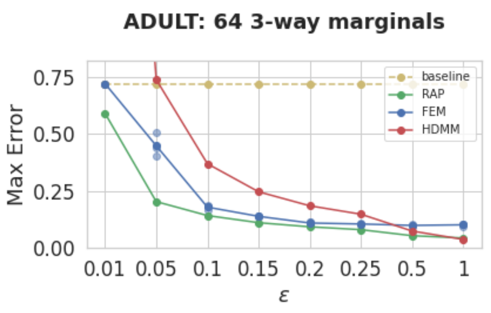
Parameter	Description	Values
K	Queries per round	5 10 25 50 100
T	Number of iterations	2 5 10 25 50

Table A2: RAP hyperparameters tested in our experiments

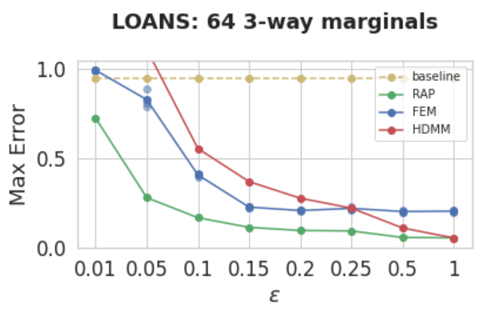
A.7 Additional Figures for Experiments

```
import jax.numpy as np
def threeway_marginals(D):
    return np.einsum('ij,ik,il->jkl', D, D, D)/D.shape[0]
```

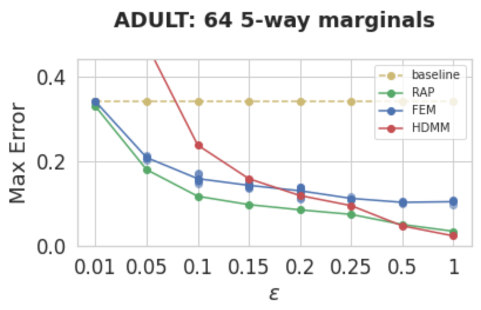
Figure A1: Python function used to compute 3-way product queries



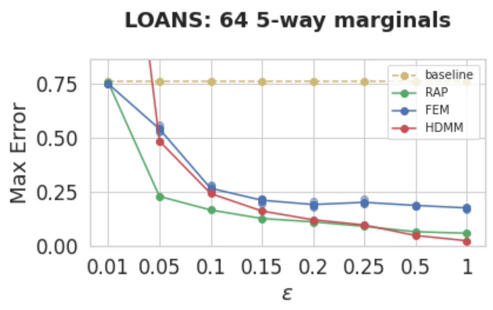
(a) ADULT dataset on 3-way marginals



(b) LOANS dataset on 3-way marginals

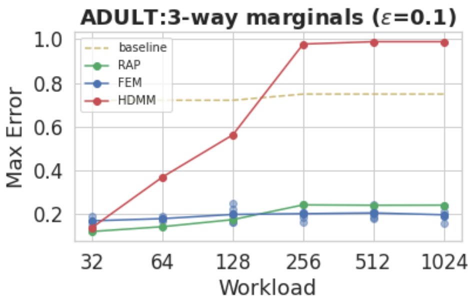


(c) ADULT dataset on 5-way marginals

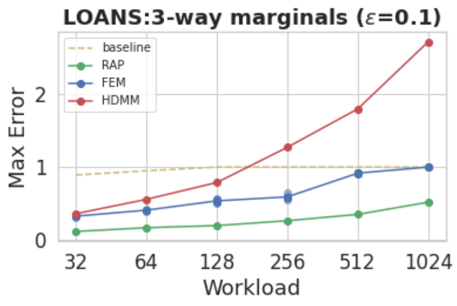


(d) LOANS dataset on 5-way marginals

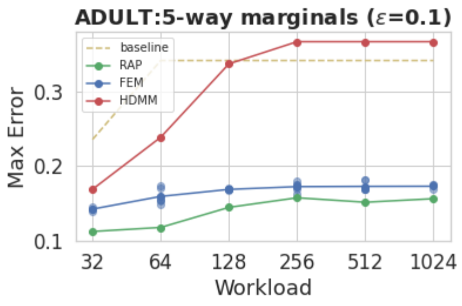
Figure A2: Max-error for 3 and 5-way marginal queries on different privacy levels. The number of marginals is fixed at 64.



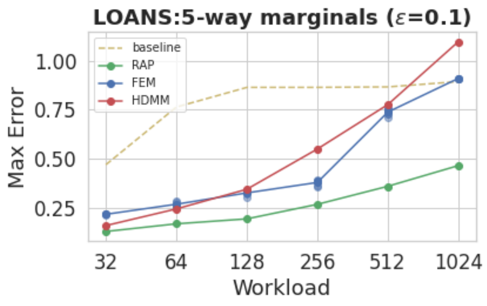
(a) ADULT dataset on 3-way marginals



(b) LOANS dataset on 3-way marginals



(c) ADULT dataset on 5-way marginals



(d) LOANS dataset on 5-way marginals

Figure A3: Max error for increasing number of 3 and 5-way marginal queries with $\epsilon = 0.1$